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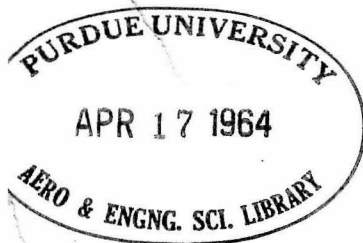
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NOTES ON STOCHASTIC PROCESSES

by
P. A. Lagerstrom

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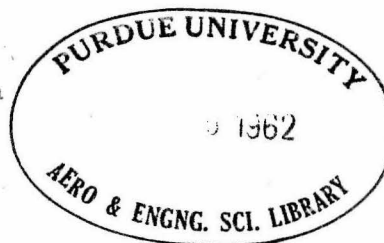




NOTES ON STOCHASTIC PROCESSES*

by

P. A. Lagerstrom



References

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1. Definition of Stochastic Process. Transition Probability

A stochastic process $X(t)$ is a function which depends on time and on chance. It bears the same relation to a definite function as a random variable does to a definite number. The "ensemble" for a random variable may be thought of as a large number of experiments, carried out under similar conditions, where each experiment provides a number. In the ensemble for a stochastic process each experiment should provide

*These notes are meant to supplement the lectures in AE 267 (1951- 1952) and are not complete.

a function of time. It is important to keep in mind that "averages" and similar statistical terms refer to averages over the ensemble. Under certain conditions the time average of one function of the ensemble may be shown to be equal to the ensemble average. It is however, necessary to distinguish clearly between the two concepts.

A precise definition of the concept of a stochastic process may be made in many ways (cf. Refs. 2 - 5). The following definition (cf. especially Ref. 3) will be sufficient for many physical applications.

Consider n -values of the time coordinate: $t_1 < t_2 \dots t_n$. $X(t)$ is said to be a stochastic process if for any such values $(X(t_1), \dots, X(t_n))$ is a random vector. The probability structure of the stochastic process is thus determined if the probability structure of each such random vector is determined.

We shall only consider the case of continuous probability where the probability is determined by a probability density. This density (frequency function f of Ref. 1) is defined in the n -dimensional space consisting of all possible values of the vector. Following Ref. 3 this density will be denoted by W_n . If " Pr " stands for the probability of an event, then

$$W_1(x_1, t_1) dx_1 = Pr \{ x_1 \leq X(t_1) \leq x_1 + dx_1 \} \quad (1.1a)$$

$$W_2(x_1, t_1; x_2, t_2) dx_1 dx_2 = Pr \{ x_1 \leq X(t_1) \leq x_1 + dx_1, \text{ and } x_2 \leq X(t_2) \leq x_2 + dx_2 \} \quad (1.1b)$$

$$W_3(x_1, t_1; x_2, t_2; x_3, t_3) = \text{etc.}$$

The concept of conditional probability (Ref. 1, page 157) plays a very important role in the theory of stochastic processes. The term "transition probability" is often used instead.

$$T_n(x_1, t_1; \dots x_{n-1}, t_{n-1} | x_n, t_n) dx_n = Pr \{x_n \leq X(t_n) \leq x_n + dx_n \} \quad (1.2)$$

under the hypothesis that $X(t_1) = x_1, \dots X(t_{n-1}) = x_{n-1}$

From elementary theory of probability (Ref. 1) various relations between the W_n and T_n may be derived such as:

$$W_1(x_1, t_1) = \int_{-\infty}^{\infty} W_2(x_1, t_1; x_2, t_2) dx_2 \quad (1.3a)$$

$$W_1(x_2, t_2) = \int_{-\infty}^{\infty} W_2(x_1, t_1; x_2, t_2) dx_1 \quad (1.3b)$$

$$W_2(x_1, t_1; x_3, t_3) = \int_{-\infty}^{\infty} W_3(x_1, t_1; x_2, t_2; x_3, t_3) dx_2 \quad (1.3c)$$

$$W_2(x_1, t_1; x_2, t_2) = W_1(x_1, t_1) T_2(x_1, t_1 | x_2, t_2) \quad (1.3d)$$

$$W_3(x_1, t_1; x_2, t_2; x_3, t_3) = W_2(x_1, t_1; x_2, t_2) T_3(x_1, t_1; x_2, t_2 | x_3, t_3) \quad (1.3e)$$

$$\int_{-\infty}^{\infty} T_2(x_1, t_1 | x_2, t_2) dx_2 = 1 \quad (1.3f)$$

Combining these one obtains

$$\begin{aligned} W_1(x_1, t_1) T_2(x_1, t_1 | x_3, t_3) &= W_2(x_1, t_1; x_3, t_3) = \\ &= \int W_3(x_1, t_1; x_2, t_2; x_3, t_3) dx_2 = \int W_2(x_1, t_1; x_2, t_2) T_3(x_1, t_1; x_2, t_2 | x_3, t_3) dx_2 \\ &= \int W_1(x_1, t_1) T_2(x_1, t_1 | x_2, t_2) T_3(x_1, t_1; x_2, t_2 | x_3, t_3) dx_2 \end{aligned}$$

Dividing the first and last terms by $W_1(x_1, t_1)$ one obtains the important composition rule for transition probabilities

$$T_2(x_1, t_1 | x_3, t_3) = \int_{-\infty}^{\infty} T_2(x_1, t_1 | x_2, t_2) T_3(x_1, t_1; x_2, t_2 | x_3, t_3) dx_2 \quad (1.4)$$

where $t_1 < t_2 < t_3$.

This rule has a direct intuitive interpretation which is left to the reader. In such interpretations it is sometimes convenient to visualize the stochastic process as a particle whose position on the

time x -axis varies according to some statistical laws (as for example in the random walk problem). $T_2(x, t, | x_3, t_3)$ is the probability density of finding the particle at x_3 at time t_3 if it has been previously observed at x , at time t .

2. Markoff Processes

The theory of stochastic processes is well developed only for certain specialized types of processes, such as Markoff processes and stationary processes. A stochastic process is said to be a Markoff process if the probability structure at $t = t_n$ is completely determined from a knowledge of the state of the process at any one previous term t_{n-1} :

$$T_n(x, t; \dots x_{n-1}, t_{n-1} | x_n, t_n) = T_2(x_{n-1}, t_{n-1} | x_n, t_n) \quad (2.1)$$

It follows that W_2 completely defines such a process. To show this W_n has to be determined from W_2 . According to (1.3a) W_2 determines W_1 . W_1 and W_2 determine T_2 by (1.3d) (disregarding the case where $W_1 = 0$). T_2 determines T_n by (2.1). W_2 and T_3 determine W_3 by (1.3e) etc.

Since only T_2 matters the subscript "2" will be dropped. The composition rule (1.4) reduces to the Chapman equation:

$$T(x, t, | x_3, t_3) = \int T(x, t, | x_2, t_2) T(x_2, t_2 | x_3, t_3) dx_2 \quad (2.2)$$

This is the fundamental integral equation for the transition probability of a Markoff process.

A Markoff process is said to be homogeneous in time if T depends only on the time difference $t_2 - t$. Similarly it is homogeneous in space if T depends only on the relative distance $x_2 - x$. In the

former case the notation $T(x, | x_2, t_2 - t,)$ will be used. Thus for a process homogeneous in time:

$$T(x, | x_2, t) = T(x, t, | x_2, t_2) \quad (2.3)$$

where t_1 and t_2 are any numbers such that $t_2 - t_1 = t > 0$. Furthermore (2.2) reduces to the Smoluchowski equation

$$T(x, | x_2, t) = \int T(x, | x_2, t') T(x_2 | x_2, t - t') dx_2 \quad (2.4)$$

where $0 < t' < t$.

Homogeneity in time means that the laws governing the process are invariant under a change of origin of the time coordinate. It does not mean that the probability structure of any given process with T as the transition probability is independent of time or that the process is stationary in the sense to be discussed later. One may say that whereas w_2 completely determines the probability structure of a Markoff process, T determines only the laws of the process. In addition to T initial conditions or some similar information is needed. For example, if T and $w_1(x, 0)$ are known w_2 is determined as follows for times larger than 0:

$$w_1(x, t) = \int w_1(x, 0) T(x | x, t) dx \quad (2.5)$$

$$w_2(x, t, | x_2, t_2) = w_1(x, t) T(x, | x_2, t_2 - t) \quad (2.6)$$

For T itself one may regard the following equation as an initial condition for the integral equations (2.2) or (2.4)

$$T(x, t | x_2, t) = \delta(x_2 - x) \quad (2.7)$$

It is left to the reader to find the trivial statement hidden beneath the fancy mathematical notation in (2.7).

3. The Kolmogoroff and Fokker-Planck Equations.

Consider a general Markoff process. The transition probability $T(x, t | y, s)$ may be regarded as a function of x, t for fixed y, s or as a function of y, s for x, t . Under certain very general conditions these functions satisfy partial differential equations related to the heat equation. In the first case the equation is called Kolmogoroff's first equation, in the second case Kolmogoroff's second equation or the Fokker-Planck equation. These equations are equivalent to the integral equation (2.2). The probability density w_t will also satisfy the second equation. A derivation of these equations will be sketched below. For further details, see Ref. 3 and in particular Ref. 4.

It will be assumed that the following limits of moments exist:

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (x' - x) T(x, t - \Delta t | x', t) dx' = a(x, t) \quad (3.1a)$$

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (x' - x)^2 T(x, t - \Delta t | x', t) dx' = b(x, t) \quad (3.1b)$$

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (x' - x)^3 T(x, t - \Delta t | x', t) dx' = 0 \quad (3.1c)$$

$a(x, t)$ may be interpreted as the rate of displacement of the mean position of the particle. If σ is the dispersion (Ref. 1, page 180) in time Δt relative to the initial position of the particle then $b(x, t) = \lim_{\Delta t \rightarrow 0} \frac{\sigma^2}{\Delta t}$. In most important applications σ^2 is actually of order Δt and b exists and is different from zero.

The Chapman equation (2.2) will now be written for the time values $t - \Delta t < t < s$:

$$T(x, t - \Delta t | y, s) = \int T(x, t - \Delta t | x', t) T(x', t | y, s) dx' \quad (3.2)$$

If the expansions

$$T(x, t - \Delta t | y, s) = T(x, t | y, s) - \Delta t \left(\frac{\partial T}{\partial t} \right) + \dots \quad (3.3a)$$

$$T(x', t | y, s) = T(x, t | y, s) + (x' - x) \frac{\partial T}{\partial x} + \frac{(x' - x)^2}{2} \frac{\partial^2 T}{\partial x^2} + \dots \quad (3.3b)$$

are inserted into (3.2) and one then lets Δt tend to zero using (3.1) and (1.3f) one obtains Kolmogoroff's first equation

$$-\frac{\partial T}{\partial t} = a(x, t) \frac{\partial T}{\partial x} + \frac{1}{2} b(x, t) \frac{\partial^2 T}{\partial x^2} \quad (3.4)$$

If the process is homogeneous in time and $t = s - t$, then

$T(x, t | y, s) = T(x, | y, s - t)$ and $\frac{\partial T}{\partial t} = -\frac{\partial T}{\partial t}$. Furthermore a and b become independent of time and (3.4) reduces to

$$\frac{\partial T}{\partial t} = a(x) \frac{\partial T}{\partial x} + b(x) \frac{\partial^2 T}{\partial x^2} \quad (3.5)$$

If the process is also homogeneous in space a and b are constants.

The second equation is derived as follows (Ref. 3, page 331):

Consider a function $R(y)$ which is arbitrary except for the obvious integrability and differentiability properties needed in the proof below. For infinitesimal values of Δs

$$\begin{aligned} \Delta s \int \frac{\partial T(x, t | y, s)}{\partial s} R(y) dy &= \int [T(x, t | y, s + \Delta s) - T(x, t | y, s)] R(y) dy \\ &= \int \left[\int T(x, t | y', s) T(y', s | y, s + \Delta s) dy' \right] R(y) dy - \int T(x, t | y, s) R(y) dy \\ &= \int \left[\int T(y', s | y, s + \Delta s) \left(R(y') + (y - y') R'(y') + \frac{(y - y')^2}{2} R''(y') + 1 \right) dy' \right] T(x, t | y, s) dy \\ &\quad - \int R(y') T(x, t | y', s) dy' = \\ &= \Delta s \int T(x, t | y', s) \left[a(y', s) R'(y') + b(y', s) R''(y') \right] dy' \end{aligned}$$

Integrating by parts under the assumption that all functions involved vanish at infinity and replacing the bound variable y' by y one obtains from the first and last terms:

$$\int \frac{\partial T}{\partial s} R(y) dy = \int \left[-\frac{\partial}{\partial y} (aT) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (bT) \right] R(y) dy \quad (3.6)$$

Since this relation is valid for an arbitrary function $R(y)$ it follows that

$$\frac{\partial T}{\partial s} = -\frac{\partial}{\partial y} (a(y, s) T) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (b(y, s) T) \quad (3.7)$$

where $T = T(x, t | y, s)$ and a and b are in general functions of y and s . (3.7) is called the Fokker-Planck equation or Kolmogoroff's second equation

If the process is homogeneous in time (3.7) reduces to

$$\frac{\partial T}{\partial \tau} = -\frac{\partial}{\partial y} (a(y) T) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (b(y) T) \quad (3.8)$$

where $\tau = s - t$.

The probability density W , also satisfies (3.7). Dropping the subscript "1" one has

$$W(y, s) = \int w(x, t) T(x, t | y, s) dx \quad (3.9)$$

By applying the differential operators of (3.7) to both sides of (3.9) one obtains the desired equation. The space and time coordinates may here be denoted by x and t since the distinction of initial coordinates versus final coordinates does not enter when W alone is considered. Thus

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x} (a(x, t) W) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b(x, t) W) \quad (3.10)$$

This is a parabolic equation similar to the heat equation. For an unbounded domain ($-\infty < x < \infty$) the solution for $t > t_0$ is completely determined by prescribing the initial values of W : $W(x, t_0)$, together with the requirement that W is finite at $x = \pm \infty$. In all probability problems one actually must require that $w(\pm \infty, t) = 0$ since $\int_{-\infty}^{\infty} w(x, t) dx = 1$.

The solution which at $t=t_0$ takes the value $w(x, t_0) = \delta(x - x_0)$ may be regarded as the fundamental solution of (3.10). From the definition of the transition probability it follows that the fundamental solution is simply $T(x_0, t_0 | x, t)$. (3.7) expresses the fact that the fundamental solution satisfies the differential equation considered. As is known from the theory of partial differential equations, the fundamental solution also satisfies the so-called adjoint equation when regarded as a function of the initial coordinates. Actually (3.4) is the adjoint of (3.8). Finally, (3.9) interprets in probability language the well-known fact that when the fundamental solution of a linear equation is known the general initial value problem is solved by a simple superposition.

It is left to the reader to specialize the general discussion of this section to the problem of random walk discussed in the lectures and to the "asymmetric" random walk as discussed in Ref. 2. The present derivation of (3.10) should also be compared with the derivation of a special case discussed in the lectures. In the latter case a discrete random walk was considered when the particle jumps $\pm \Delta x$ at intervals Δt . By a limiting process the heat equation was obtained. Note how the analogue of (3.1a,b) is used in this derivation.

NOTES ON STOCHASTIC PROCESSES (2)*

4. Review of Harmonic Analysis

Additional References

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11. J. Lawson and G. E. Uhlenbeck. Threshold Signals. M.I.T. Radiation Laboratory, Series 24.
12. G. C. Clementson: An Investigation of the Power Spectral Density of Atmospheric Turbulence. M.I.T. Instrumentation Laboratory, 6445-T-31, May 1950.

Some concepts and theorems from harmonic analysis will be reviewed below. The formulation and proofs of the theorems will not aim at mathematical rigor or completeness. For a mathematical treatment of the subject see in particular Ref. 7, Ref. 8 (the latter does not treat generalized harmonic analysis) and Ref. 5 and Ref. 9. For applications to stochastic problems in Physics see Refs. 5, 6, and 9b-12.

It will always be assumed that any function $f(t)$ occurring in the formula below is such that $|f(t)|$ and $|f(t)|^2$ have finite integrals over any finite interval. Whether these integrals remain finite or not when the domain of integration becomes infinite will be discussed

*Cf. footnote on page 1.

explicitly for each case. The distinction between continuous and differentiable functions will be irrelevant for most applications and hence often disregarded. On the other hand discontinuous functions will play an important role. The functions considered may be complex-valued. *The complex of $f(t)$ will be denoted by $\overline{f(t)}$.*

A. Classical Harmonic (Fourier) Analysis

For any function $f(t)$, the truncated function $f_T(t)$ is defined as follows

$$\begin{aligned} f_T(t) &= f(t) & |t| < T \\ f_T(t) &= 0 & |t| > T \end{aligned} \quad (4.1)$$

For each frequency ω and interval $-T < t < T$ a Fourier coefficient $a_T(\omega)$ is assigned to f by the definition

$$a_T(\omega) = \int_{-T}^T e^{-i\omega t} f(t) dt = \int_{-\infty}^{\infty} e^{-i\omega t} f_T(t) dt \quad (4.2)$$

It is of great interest for the physical applications (in particular to stochastic processes) to classify functions according to the behavior of $a_T(\omega)$ as $T \rightarrow \infty$. The two cases studied in classical Fourier analysis are the following (the third case will be discussed in Section D):

(1) Almost Periodic Functions. These are represented by the form

$$f(t) = \sum A_n e^{i\omega_n t} \quad (4.3)$$

A_n is called the complex amplitude of the ω_n -component. If we write $A_n = R_n e^{i\varphi_n}$, $R_n > 0$ then R_n is the (real) amplitude and φ_n the phase. If all frequencies ω_n are multiples of a basic frequency, then $f(t)$ is periodic. Note that an almost periodic function does not die out as $t \rightarrow \infty$ but keeps on oscillating. Hence $|f(t)|$ and $|f(t)|^2$ do not have finite integrals over an infinite interval. The oscillations

are "persistent" and ^{also} regular in the sense that they may be decomposed into discrete harmonic components. In this case $a_T(\omega)$ does not approach a finite limit as $T \rightarrow \infty$. For any ω_n occurring in (4.2) the rate of increase of $a_T(\omega_n)$ is of the order of T . More precisely

$$\begin{aligned} \frac{a_T(\omega)}{2T} &= A_n & \omega &= \omega_n \\ \lim_{T \rightarrow \infty} \frac{a_T(\omega)}{2T} &= 0 & \omega &\neq \omega_n \text{ ^{all any} } \end{aligned} \quad (4.4a)$$

If $f(t)$ has the period $2T_0$ then

$$\lim_{T \rightarrow \infty} \frac{a_T(\omega_n)}{2T} = \frac{a_{T_0}(\omega_n)}{2T_0} \quad (4.4b)$$

(2) Functions which Die Out Sufficiently Strongly at Infinity.

By "sufficiently strongly" is meant that $|f(t)|$ and $|f(t)|^2$ have a finite integral over the whole infinite interval. In this case $a_T(\omega)$ will tend to a finite limit as $T \rightarrow \infty$, which will be denoted by $a(\omega)$. (4.4) and (4.3) are then replaced by

$$\lim_{T \rightarrow \infty} a_T(\omega) = a(\omega) \quad (4.5)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} a(\omega) d\omega \quad (4.6)$$

(4.6) may be regarded as a limiting case of (4.3) in the following sense. Consider a function which within the interval $(-T, T)$ coincides with $f(t)$ and repeats itself periodically outside this interval. This periodic function may then be expressed in the form (4.3) where $\omega_n = n \cdot \Delta\omega$, and $\Delta\omega = \frac{2\pi}{2T}$ is the basic frequency associated with the interval. The A_n may be computed as

$$A_n = \frac{1}{2T} \int_{-T}^T e^{-i\omega_n t} f(t) dt = \frac{a_T(\omega_n)}{2T}$$

Hence for $-T < t < T$

$$f(t) = \sum A_n e^{i\omega_n t} = \sum \frac{a_T(\omega_n)}{2T} e^{i\omega_n t} = \frac{1}{2\pi} \sum a_T(\omega_n) e^{i\omega_n t} \Delta\omega \quad (4.7)$$

As $T \rightarrow \infty$, $\Delta\omega$ tends to zero, the expression above will represent $f(t)$ inside an increasingly larger interval and the sum will tend to the integral (4.6).

Note that for $-T < t < T$, $f(t)$ coincides with $f_T(t)$ which may be represented as a Fourier integral:

$$f_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a_T(\omega) e^{i\omega t} d\omega \quad (= f(t) \text{ for } |t| < T) \quad (4.8)$$

Thus when only a finite interval $(-T, T)$ is given it does not make sense to talk about the exact Fourier coefficients associated with $f(t)$.

But one may say that the Fourier coefficient for the frequency band $(\omega_n - \frac{\Delta\omega}{2} < \omega < \omega_n + \frac{\Delta\omega}{2})$ is $\frac{a_T(\omega_n)}{2T}$. This is obvious for the series representation (4.7). For the integral representation (4.8) note that

$$\int_{\omega_n - \frac{\Delta\omega}{2}}^{\omega_n + \frac{\Delta\omega}{2}} \frac{a_T(\omega)}{2\pi} d\omega$$

is approximately $\frac{a_T(\omega_n)}{2T}$ if $a_T(\omega)$ does not vary much in the frequency band considered. Since $\Delta\omega = \frac{2\pi}{2T}$, the band width may be narrowed by increasing the length of the interval considered. A consequence is that no matter how perfect measuring instruments are used, if the time of an experiment is $2T$, it is a priori impossible to have a resolution of the frequencies finer than $\frac{2\pi}{2T}$. One may also use the following terminology: $\frac{a_T(\omega)\Delta\omega}{2\pi}$ is the complex amplitude assigned to the frequency band $(\omega - \frac{\Delta\omega}{2}, \omega + \frac{\Delta\omega}{2})$. As the width of the band ($\Delta\omega$) decreases to zero this amplitude may tend to a constant A_n for certain ω_n in which case $f(t)$ has the periodic component $A_n e^{i\omega_n t}$ or it may tend to zero. In the latter case the ω -component of $f(t)$ has

infinitesimal amplitude $\frac{a(\omega)}{2\pi} d\omega$ and $a(\omega)$ represents the amplitude density at ω .

Both the Fourier series (4.3) and the Fourier integral (4.6) may be written as a Fourier-Stieltje's integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} dF(\omega) \quad (4.9)$$

For the Fourier series $F(\omega)$ is a step function with jumps $2\pi A_n$ at ω_n , i.e. a function which is constant for $\omega_n < \omega < \omega_{n+1}$ and such that $F(\omega_n + 0) - F(\omega_n - 0) = 2\pi A_n$. For the Fourier integral $F'(\omega) = a(\omega)$. If $F(\omega)$ has discontinuities but varies in a differentiable way in between (4.9) represents a function which is a sum of the two types. It will be seen later that the correlation function of a stationary stochastic process and the autocorrelation of a function as defined in Section D may be expressed in the form (4.9). On the other hand the classical Fourier analysis described above is insufficient for many stochastic processes of great physical interest. This will be discussed later on in Section D.

B. Some Properties of Almost Periodic Functions

As was seen above the Fourier coefficients A_n of a Fourier series may be obtained as limits of a time average

$$A_n = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-i\omega_n t} f(t) dt$$

Another time average or "mean value" of great importance is the autocorrelation $\psi(\tau)$. It is defined as

$$\psi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{f(t)} f(t+\tau) dt \quad (4.10a)$$

If $f(t)$ is a Fourier series then

$$\begin{aligned}\psi(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\sum \bar{A}_n e^{-i\omega_n t} \right) \left(\sum A_n e^{i\omega_n(t+\tau)} \right) dt \\ &= \sum |A_n|^2 e^{i\omega_n \tau}\end{aligned}\quad (4.10b)$$

Thus the autocorrelation of an almost periodic function is also almost periodic with the same frequencies. If $\psi(\tau)$ is known the frequencies and associated real amplitudes of $f(t)$ are determined whereas the phases are undetermined. The coefficients $|A_n|^2$ are called the power spectrum (sometimes spectrum or energy spectrum of $f(t)$). An almost periodic function is said to have a discrete spectrum since only a discrete set of frequencies (ω_n) occur.

The correlation function $\psi(\tau)$ belongs to the class of positive definite functions. This concept is of great importance for the mathematical theory. The reader is referred to Ref. 8, page 74. Here it will only be mentioned that for real $f(t)$ this implies that $\psi(\tau)$ is real and even in τ which may be easily checked from (4.10b). Reality of $f(t)$ means that $\bar{A}_n = A_{-n}$ if ω_{-n} is defined as $-\omega_n$.

If a second almost periodic function $g(t)$ is given it may always be referred to the same frequencies as $f(t)$: $g(t) = \sum B_n e^{i\omega_n t}$ where possibly some A_n or B_n are zero. One may then form the

$$\text{Cross Correlation} = \text{Time Average of } \bar{f}(t) g(t+\tau) = \sum \bar{A}_n B_n e^{i\omega_n \tau} \quad (4.11)$$

The frequencies occurring with non-zero amplitude in (4.11) are those which are common to both $f(t)$ and $g(t)$.

C. Some Properties of Fourier Transforms

In this section functions expressible as Fourier integrals will be discussed. The relation

$$a(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

and the dual relation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} a(\omega) d\omega$$

will be expressed as " $f(t)$ has the Fourier transform $a(\omega)$ " or

" $f(t) \leftrightarrow a(\omega)$ ". The following properties then follow immediately from the definition:

$$f(t) \leftrightarrow a(\omega) \text{ implies}$$

$$f(t+\tau) \leftrightarrow e^{i\omega\tau} a(\omega) \quad (4.12a)$$

$$f(-t) \leftrightarrow a(-\omega) \quad (4.12b)$$

$$\overline{f(t)} \leftrightarrow \overline{a(-\omega)} \quad (4.12c)$$

$$f'(t) \leftrightarrow i\omega a(\omega) \quad (4.12d)$$

Thus if $f(t)$ is real then $a(-\omega) = \overline{a(\omega)}$. If in addition f is even in t , then $a(-\omega) = \overline{a(\omega)} = a(\omega)$; if f is real and odd, then $a(-\omega) = \overline{a(\omega)} = -a(\omega)$.

Formula (4.12d) may be generalized. Consider a linear differential operator with constant coefficients, i.e. a polynomial in $\frac{d}{dt}$:

$$P\left(\frac{d}{dt}\right) = \sum_{\ell=0}^n c_{\ell} \left(\frac{d}{dt}\right)^{\ell} \quad c_{\ell} = \text{constant}$$

Then

$$P\left(\frac{d}{dt}\right) f(t) = \sum_{\ell=0}^n c_{\ell} f^{(\ell)}(t) \leftrightarrow P(i\omega) a(\omega) = a(\omega) \sum_{\ell=0}^n c_{\ell} (i\omega)^{\ell} \quad (4.12e)$$

The convolution or "Faltung" of two functions $f(t)$ and $g(t)$ will be denoted by $f * g$ and is defined as

$$f * g = \int_{-\infty}^{\infty} f(s) g(t-s) ds \quad (4.13a)$$

Replacing s by $r = t - s$ proves

$$f * g = g * f = \int_{-\infty}^{\infty} f(t-r) g(r) dr \quad (4.13b)$$

The following convolution theorem will find repeated application

$$f \leftrightarrow a(\omega), \quad g \leftrightarrow b(\omega) \text{ implies } f * g \leftrightarrow a(\omega) b(\omega) \quad (4.14a)$$

Proof: From (4.13b) and (4.12a)

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} e^{-i\omega r} a(\omega) d\omega \right) g(r) dr = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} a(\omega) \left(\int_{-\infty}^{\infty} e^{-i\omega r} g(r) dr \right) d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} a(\omega) b(\omega) d\omega \end{aligned}$$

By using (4.12) one may obtain several variants of (4.14a), e.g.

$$\int_{-\infty}^{\infty} \overline{f(s)} g(t+s) ds \leftrightarrow \overline{a(\omega)} b(\omega) \quad (4.14b)$$

One may also define multiple convolutions, e.g. the triple convolution

$$f * g * h = \int_{-\infty}^{\infty} f(t-s-u) g(s) h(u) ds du \quad (4.15)$$

By writing $f(t-s-u)$ as a Fourier integral one proves the analogue of (4.14a):

$$f \leftrightarrow a(\omega), \quad g \leftrightarrow b(\omega), \quad h \leftrightarrow c(\omega) \text{ implies } f * g * h \leftrightarrow a(\omega) \cdot b(\omega) \cdot c(\omega) \quad (4.16)$$

This could also have been proved from (4.14a) by observing that $f * g * h$ is equal to $(f * g) * h$, i.e. may be formed by first taking the convolution of f and g and then the convolution of $(f * g)$ with h .

D. Generalized Harmonic Analysis (Refs. 5, 6, 7, 9, 10, 12)

In many physical problems such as white light, noise, shot effect, turbulence, one encounters functions which exhibit persistent irregular oscillations, i.e. the functions do not die out as t increases but also they are in general not sums of discrete periodic components. The

classical methods of harmonic analysis as described in Section A do not apply to such functions. Various efforts by physicists to extend the methods of harmonic analysis to such functions culminated in Wiener's mathematical theory of generalized harmonic analysis (see in particular Ref. 7, Chapter IV). The physical processes mentioned are of stochastic nature and the generalized harmonic analysis is intimately tied up with theory of stationary stochastic processes. In this section, however, it will be assumed that the functions considered are definite rather than stochastic functions.

From the examples of "irregular oscillations" given above it is clear that one may not require that the functions have finite integrals over an infinite time domain. On the other hand it will be required that the amplitudes of the oscillations do not become unbounded for large t ; divergent oscillations are excluded. Then one may expect that the integral of the absolute value of such a function is roughly proportional to the length of the interval so that the time average stays finite as the length of the interval becomes infinite. These intuitive considerations are formalized by requiring the following quantities to be finite

$$\text{Time Average} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)| dt < \infty \quad (4.17a)$$

$$\text{Autocorrelation} = \psi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{f(t)} f(t+\tau) dt < \infty \quad (4.17b)$$

The autocorrelation and its Fourier transform (power spectrum) are the central concepts in generalized harmonic analysis. Only the properties of the original function $f(t)$ which are determined by the autocorrelation are considered.

Assume now that the limits (4.17) exist. Furthermore assume that $\psi(\tau)$ is continuous at $\tau=0$. This restriction implies that ψ is continuous everywhere. (According to Ref. 5, page 40, this does not seem to exclude any functions $f(t)$ of physical interest, cf. however the discussion of (4.22) below). It may then be shown (Ref. 7 or 5) that $\psi(\tau)$ is positive definite which according to Ref. 8 is equivalent to saying that it may be expressed as a Fourier-Stieltje's integral

$$\psi(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} dF(\omega)$$

If we assume that F is differentiable except at the points of discontinuity (which again does not seem to exclude physically significant cases, cf. Ref. 5, page 44) then this Fourier-Stieltje's integral is a sum of a Fourier series and a Fourier integral

$$\psi(\tau) = \sum P_n e^{i\omega_n \tau} + \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\omega) e^{i\omega\tau} d\omega \quad (4.18)$$

The coefficients P_n associated with ω_n are called the discrete power spectrum of $f(t)$ and $p(\omega)$ the continuous power spectrum or power spectral density. Its relation to the Fourier analysis of the original function $f(t)$ will now be discussed.

The functions considered in classical harmonic analysis evidently satisfy the requirements (4.17). If $|f|$ and $|f|^2$ have finite integrals (case of the Fourier integral) the averages (4.17) are zero. In this case the autocorrelation, being zero, yields no information whatsoever about the original function, hence there is no point in considering this case. The autocorrelation for an almost periodic function was discussed in Section B. In this case the harmonic analysis of the autocorrelation yields information about the frequencies and the absolute value of the amplitudes of the Fourier components of the original

function but not about the phase relations. In many problems this is the physically significant information; the phases may lack physical significance or be undetermined anyway (cf. later sections on stationary stochastic processes).

However (4.17) may be valid for other functions than those discussed in Sections A-C. Consider the case when the correlation function exists and is a pure Fourier integral which is not identically zero. Then the original function $f(t)$ cannot be a Fourier integral (zero autocorrelation) or a Fourier series (almost periodic autocorrelation) or a sum of such functions. By analogy with the Fourier series case one may say that the original function $f(t)$ does not have discrete periodic components but that continuous bands of frequencies occur, each frequency ^{of} infinitesimal amplitude. However, the "degree of infinitesimality" has to be such that $f(t)$ does not die out for large $|t|$, otherwise $\psi(\tau)$ would be zero. It then seems plausible that $f(t)$ belongs to the class of functions with irregular but persistent oscillations.

To gain further insight into the nature of $f(t)$ consider the Fourier transform $a_T(\omega)$ of the truncated function. If $f(t)$ is to have a continuous power spectrum, $a_T(\omega)$ can neither tend to a finite limit or be proportional to T as T tends to infinity (this would imply zero or discrete power spectrum respectively). Actually $a_T(\omega)$ grows as \sqrt{T} or more precisely one may prove: If $f(t)$ has a continuous power spectrum $p(\omega)$ then

$$\lim_{T \rightarrow \infty} \frac{|a_T(\omega)|^2}{2T} = p(\omega) \quad (4.19)$$

Note, that this does not imply that the phase of $a_T(\omega)$ tends to a

limit, however, the real amplitude $|a_T(\omega)|$ must become proportional to \sqrt{T} for T large. The amplitude of any particular oscillation of frequency ω is infinitesimal in the sense that $\frac{a_T(\omega)}{2T}$ tends to zero.

However the "degree of infinitesimality" is such that it tends to zero only as $\frac{1}{\sqrt{T}}$. (If it had approached zero faster, namely as $\frac{1}{T}$, $f(t)$ would have been a Fourier integral and would have died out for $t \rightarrow \pm \infty$).

To prove (4.19) one observes that for T much larger than τ the integral $\frac{1}{2T} \int_{-T}^T \overline{f(t)} f(t+\tau) dt$ is practically equal to $\frac{1}{2T} \int_{-\infty}^{\infty} \overline{f_T(t)} f_T(t+\tau) dt$.

Hence by the convolution theorem in the form (4.14b) its Fourier transform is $\frac{\overline{a_T(\omega)} \cdot a_T(\omega)}{2T}$. Passing to the limit one obtains

$$\psi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{f(t)} f(t+\tau) dt \leftrightarrow \lim_{T \rightarrow \infty} \frac{|a_T(\omega)|^2}{2T} \quad (4.19')$$

which is equivalent to (4.19). Note that if the power spectrum also had a discrete part, then for a frequency ω_n occurring in this part with amplitude P_n , the quantity $\frac{|a_T(\omega)|^2}{2T}$ would be equal to $2T \cdot P_n$ for T large. Experimentally the discrete spectrum should then show up as peaks in a continuous power spectrum. The amplitude of the peaks should increase proportional to the time of measurement and theoretically approach infinity for an infinite interval. The presence of such peaks would indicate finite periodic components in the original function $f(t)$ ("hidden periodicities").

The abstract notion of a function whose $|a_T(\omega)|$ increases as \sqrt{T} will be illustrated later by the concrete example of the shot effect.

Generalized harmonic analysis may be applied simultaneously to several functions. Let, e.g., $f(t)$ and $g(t)$ be two functions where $f_T(t) \leftrightarrow a_T(\omega)$ and $g_T(t) \leftrightarrow b_T(\omega)$. If the cross correlation (cf. 4.11)

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{f(t)} g(t+\tau) dt \quad (4.20a)$$

exists and has a continuous spectrum its Fourier transform is

$$\lim \frac{\overline{a_T(\omega)} b_T(\omega)}{2T} \quad (4.20b)$$

In particular let $g(t)$ be the derivative $f'(t)$. If $f(t)$ has autocorrelation $\psi(\tau) \leftrightarrow p(\omega)$

$$(\text{Cross correlation of } f(t) \text{ and } f'(t)) = \psi'(\tau) \leftrightarrow i\omega p(\omega) \quad (4.21a) \quad \checkmark$$

$$(\text{Autocorrelation of } f'(t)) = -\psi''(\tau) \leftrightarrow \omega^2 p(\omega) \quad (4.21b) \quad \checkmark$$

Similar formulas are valid for the discrete spectrum.

Proof: Let $f_T(t) \leftrightarrow a_T(\omega)$. Then $f_T(t) \leftrightarrow i\omega a_T(\omega)$ by (4.12d).

Hence the spectrum of the cross correlation of f and f' is

$$\lim \frac{\overline{a_T(\omega)} \cdot i\omega a_T(\omega)}{2T} = i\omega \lim \frac{\overline{a_T(\omega)} a_T(\omega)}{2T} = i\omega p(\omega)$$

By (4.12d) this is the Fourier transform of $\psi'(\tau)$. Similarly the power spectrum of $f'(t)$ is

$$\lim \frac{\overline{i\omega a_T(\omega)} \cdot i\omega a_T(\omega)}{2T} = \omega^2 p(\omega) \quad \checkmark$$

which by (4.12e) is the Fourier transform of $-\psi''(\tau)$. If

$$f(t) = \sum A_n e^{i\omega_n t}$$

then

$$f'(t) = \sum i\omega_n A_n e^{i\omega_n t}$$

the cross correlation is

$$\sum i\omega_n |A_n|^2 e^{i\omega_n \tau}$$

and the autocorrelation of $f'(t)$ is

$$\sum \omega_n^2 |A_n|^2 e^{i\omega_n \tau} = -\frac{d^2}{d\tau^2} \sum |A_n|^2 e^{i\omega_n \tau} \quad \checkmark$$

However, the case may occur when, e.g., $\psi(\tau)$ exists and has the Fourier

transform $p(\omega)$ but $\omega^2 p(\omega)$ is not integrable and hence not a Fourier transform of any function. In this case the original function $f(t)$ is said to be non-differentiable. An example is the following correlation function (cf. Ref. 3)

$$\psi(\tau) = e^{-|\tau|} \longleftrightarrow \frac{2}{1+\omega^2} \quad (4.22)$$

Obviously the integral

$$\int_{-A}^A e^{-i\omega\tau} \frac{2\omega^2}{1+\omega^2} d\omega$$

does not approach any finite limit as A tends to infinity. Hence no function has the Fourier transform $\frac{2\omega^2}{1+\omega^2}$. This is related to the fact that $\psi(\tau)$ has a discontinuity at $\tau=0$ and hence $\psi(\tau)$ is infinite. The spectrum $\frac{2\omega^2}{1+\omega^2}$ may still be used if one applies a "cut-off" for high frequencies, i.e. defines it to be zero for ω sufficiently large. This corresponds essentially to modifying $\psi(\tau)$ for small values of τ .

E. Linear Differential Equations with Constant Coefficients

This section will review some basic facts about the harmonic analysis of solutions to linear differential equations with constant coefficients

$$\mathcal{L}(X) = f(t) \quad (4.23a)$$

where X is a function of t and \mathcal{L} is a polynomial of the time derivative:

$$\mathcal{L} = P\left(\frac{d}{dt}\right) = \sum_{\ell=0}^n c_{\ell} \frac{d^{\ell}}{dt^{\ell}} \quad c_{\ell} = \text{constant} \quad (4.23b)$$

$X(t)$ will be called the response to the forcing function $f(t)$ or the output resulting from the input $f(t)$. It will be assumed that the initial conditions are zero. This defines $X(t)$ uniquely when $f(t)$ is zero for $t < \text{some } t_0$. In the general case $X(t)$ may be defined as the

limit of the response to $f_T(t)$ as $T \rightarrow \infty$.

The response to a unit impulse at $t=0$ will be denoted $h(t)$.

It plays the role of a fundamental solution of (4.23):

$$\mathcal{L}(h) = \delta(t) \quad \delta(t) = \text{Dirac delta-function} \quad (4.24)$$

The solution of (4.23) for a general $f(t)$ is then

$$X(t) = \int_{-\infty}^t f(s) h(t-s) ds = \int_{-\infty}^{\infty} f(s) h(t-s) ds = f * h = \int_{-\infty}^{\infty} f(t-s) h(s) ds \quad (4.25)$$

It will be assumed that \mathcal{L} represents a system with damping so that $h(t)$ dies out exponentially as $t \rightarrow \infty$. In this case $h(t)$ has a Fourier transform $b(\omega)$. By multiplying (4.24) by $e^{-i\omega t}$, integrating from $t = -\infty$ to ∞ and using (4.24) one finds that $b(\omega) = \frac{1}{P(i\omega)}$. On the other hand it follows by standard methods that $\frac{e^{i\omega t}}{P(i\omega)}$ is the response to the simple harmonic forcing function $e^{i\omega t}$. Hence

$h(t)$ = response to unit impulse at $t=0$ has Fourier transform $b(\omega) = \frac{1}{P(i\omega)}$ = complex amplitude of response (4.26) to $e^{i\omega t}$ or (output/input) for simple harmonic input

$P(i\omega)$ is called the impedance and $P(i\omega)^{-1}$ the admittance or transfer function. (4.26) shows that the condition for damping is that $P(i\omega)$ is never zero for real values of ω .

If $f(t)$ has a Fourier transform $a(\omega)$ then (4.25) and the convolution theorem (4.14a) imply that

$$X(t) \leftrightarrow \frac{a(\omega)}{P(i\omega)} \quad (4.27)$$

Of greater interest is the case when $f(t)$ does not die out at $t \rightarrow \infty$ and has a non-vanishing power spectrum. If $f(t)$ is almost periodic equal to $\sum A_n e^{i\omega_n t}$ an elementary calculation (cf. last definition

of transfer function in (4.26)) shows that $X(t) = \sum C_n e^{i\omega_n t}$, $C_n = \frac{A_n}{P(i\omega_n)}$. In particular the power spectrum of $X(t)$ is discrete with the same frequencies as that of $f(t)$ and amplitude given by

$$\begin{aligned} \text{Power spectrum of } X(t) &= |C_n|^2 = \left| \frac{A_n}{P(i\omega_n)} \right|^2, \text{ where} \\ |A_n|^2 &= \text{power spectrum of } f(t) \end{aligned} \quad (4.28)$$

If $f(t)$ has a continuous power spectrum $p(\omega)$, define $X_T(t)$ as the response to the truncated forcing function $f_T(t)$:

$$X_T = f_T * h \quad (4.29)$$

X_T is not quite the truncated version of $X(t)$ = response to $f(t)$ since X_T may not vanish for $t > T$. However, due to the assumed damping this difference will be irrelevant in the limiting procedures below. Let $f_T(t) \leftrightarrow a_T(\omega)$. Then the convolution theorem and (4.29) (cf. also (4.27)) imply that $X_T \leftrightarrow \frac{a_T(\omega)}{P(i\omega)}$. Hence

$$\begin{aligned} \text{Power spectrum of } X(t) &= \lim_{T \rightarrow \infty} \left| \frac{a_T(\omega)}{P(i\omega)} \right|^2 \cdot \frac{1}{2T} = \frac{p(\omega)}{|P(i\omega)|} \\ \text{where } p(\omega) &= \text{power spectrum of } f(t) \end{aligned} \quad (4.30)$$

This formula is the analogue of (4.28) for the continuous case.

If $\psi(\tau)$ and $\varphi(\tau)$ are the autocorrelations of $X(t)$ and $f(t)$ respectively, they must be connected by a relation dual to (4.30). In fact

$$\begin{aligned} \psi(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{X(t)} X(t+\tau) dt = \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{f(t-s)} h(s) f(t+\tau-u) h(u) ds du dt = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\tau-s-u) \overline{h(s)} h(u) \end{aligned}$$

The last integral is related to the triple correlation (4.15).

Hence (4.16) provides another proof of (4.30) similar to the proof of (4.19').